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(2, 6)-cages and their spectra

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Dedicated to A.A. Zykov on the occasion of his 90th birthday

July 5, 2012

Abstract

A $(k, 6)$-cage $(k = 2, 3, 4, 5)$ is a 2-connected cubic plane graph that has only k-gons and hexagons as its faces. Continuing their work on $(3, 6)$-cages (2009) the authors investigate the combinatorial (topological and algebraic) structure of $(2, 6)$-cages and explicitly determine their eigenvalues and eigenvectors.

Keywords: cubic graph, $(k, 6)$-cage, polyhedron, toroidal 6-cage; eigenvalue, eigenvector; perfect matching.

Introduction

A *pseudo-polyhedron* is the embedding of a finite vertex-2-connected planar graph that may have multiple edges but no loops in the (topological) sphere. A $(k, 6)$*-cage* is a cubic (i.e., trivalent) pseudo-polyhedron that has only k-gons and hexagons as its faces. By Euler's polyhedron formula, $(k, 6)$-cages can exist only for $k = 2, 3, 4, 5$. The number of vertices of any $(k, 6)$-cage is even. The only $(k, 6)$-cage on two vertices is the 2-cage Δ_2 (Figure 1). Δ_2 is edge-3-connected but not vertex-3-connected. A

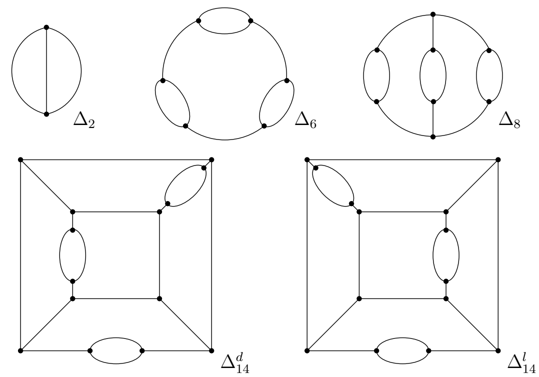


Figure 1. The smallest five $(2,6)$-cages. Note the chirality of Δ_{14}^d and Δ_{14}^l (d = dextro, l = laevo; the subscript counts the vertices)

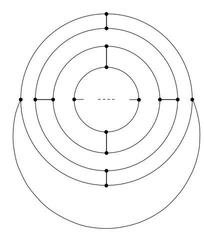


Figure 2. The class of edge- and vertex-2-connected $(3,6)$-cages

$(k,6)$-cage different from Δ_2 that is edge-3-connected is also vertex-3-connected, and conversely. $(2,6)$-cages with more than two vertices are not edge-3-connected (Figure 1). For $k = 4, 5$ a $(k,6)$-cage is vertex-3-connected, thus has a realization as a polyhedron. $(3,6)$-cages depend on three parameters [7], [9]; except for those of a 1-parametric subclass (Figure 2), $(3,6)$-cages, too, are vertex-3-connected.

The structure of $(3,6)$-cages is well known [6], [7], [9].

A *toroidal* 6-*cage* is (the graph of) a cubic hexagonal tessellation of the torus (Figure 3).

Much attention has been attracted by $(5,6)$-cages (Figure 4) and toroidal 6-cages (Figure 3) because the combinatorial structure of a polycyclic pure carbon molecule C_n (spherical or toroidal fullerene) is reflected by such a cage.

The fact that there are many relations between toroidal 6-cages and $(k,6)$-cages, and between $(k,6)$-cages for different values of k, justifies the intensive study of all of these types of cages.

In [9] spectra and eigenvectors of toroidal 6-cages and $(3,6)$-cages were determined. Continuing those considerations, we shall in Part I of this paper investigate

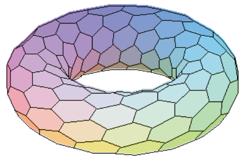


Figure 3. A toroidal 6-cage

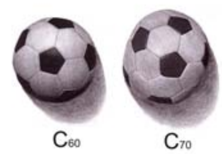


Figure 4. $(5, 6)$-cages

the structure of $(2, 6)$-cages and in Part II calculate their spectra and eigenvectors.

Part I. The structure of $(2, 6)$-cages

I. 1 The topological structure

By Euler's formula, the number of digons in a $(2, 6)$-cage is three (Figure 1).

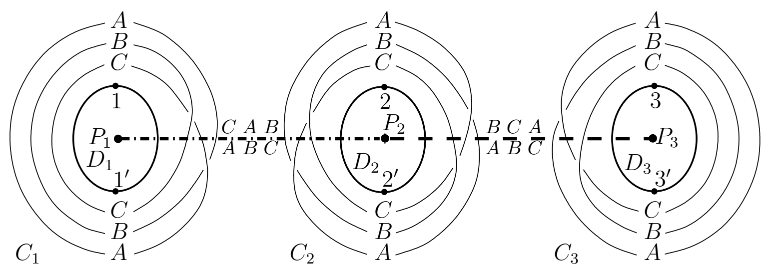
Let Δ with graph $\mathbb{D}$ be a $(2, 6)$-cage with vertices $v_1, v_2, \ldots, v_n$ and digons D_1, D_2, D_3; fix a point P_i in the interior of D_i $(i = 1, 2, 3)$. Let $\mathbf{A}, \mathbf{B}, \mathbf{C}$ be three identical copies of the plane with Δ drawn on it. Make in $\mathbf{A}, \mathbf{B}, \mathbf{C}$ identical cuts c_{12} and c_{23} from P_1 to P_2 and from P_2 to P_3, respectively, and, as indicated in Figure 5(a), following the cuts glue planes $\mathbf{A}, \mathbf{B}, \mathbf{C}$ together, thus turning the boundaries of the digons D_i into circuits C_i each of length 6.

This results in an oriented closed surface $\mathbb{T}$ with a vertex-labelled cubic graph $\mathbb{G} = \mathbb{G}(\Delta)$ embedded in it where every label ν $(\nu = 1, 2, \ldots, n)$ occurs at precisely three distinct vertices of the embedding $\Gamma = \Gamma(\Delta)$ of $\mathbb{G}$. Recall that, by Euler's formula, the torus is the only oriented closed surface that admits of cubic hexagonal tessellations. All faces of Γ being hexagons, $\mathbb{T}$ must be the torus (Figure 5(b)), and Γ is a specifically vertex-labelled toroidal 6-cage.

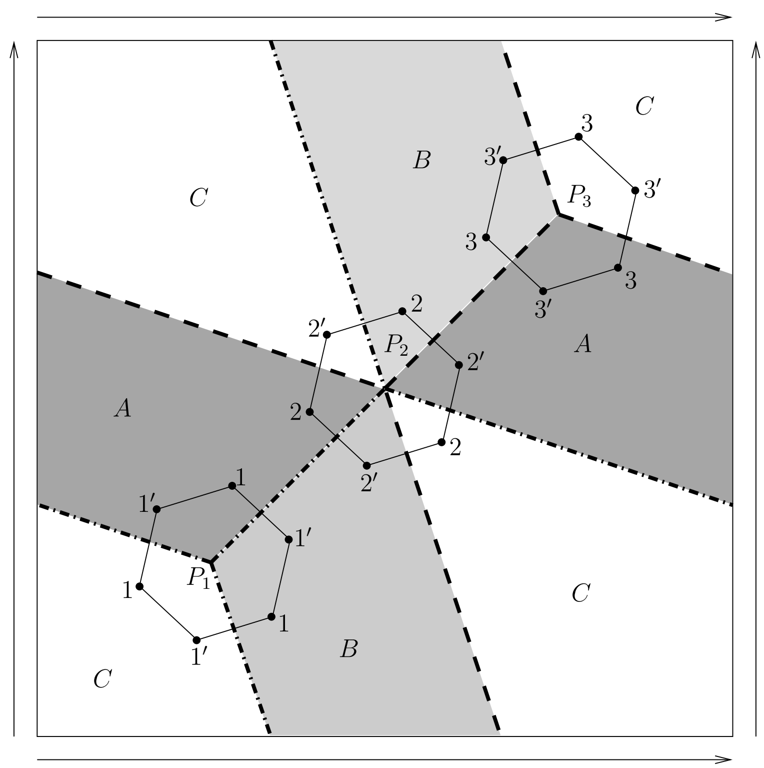
The labelling being neighbourhood preserving, $\mathbb{D}$ is a divisor[a] of $\mathbb{G}$, or, equivalently, $\mathbb{G}$ is a cover of $\mathbb{D}$.

Let $\mathfrak{H}$ denote the regular hexagonal tessellation of the plane (the graphite sheet). Torus $\mathbb{T}$, with Γ drawn on it, can be represented by a parallelogram $\mathbb{P}$ as a part of $\mathfrak{H}$ with boundary identification [1], and this representation can be extended over the whole plane such that the labelling becomes a twofold periodic function defined on the vertices of $\mathfrak{H}$, with fundamental paralellogram $\mathbb{P}$ (Figure 6; cf. [9]). Let Γ^*

[a] For the divisor concept see, e.g., [5].



(a) Construction of the surface $\mathbb{T}$



(b) The surface $\mathbb{T}$ is a torus

Figure 5. Representing a $(2,6)$-cage as a labelled toroidal 6-cage

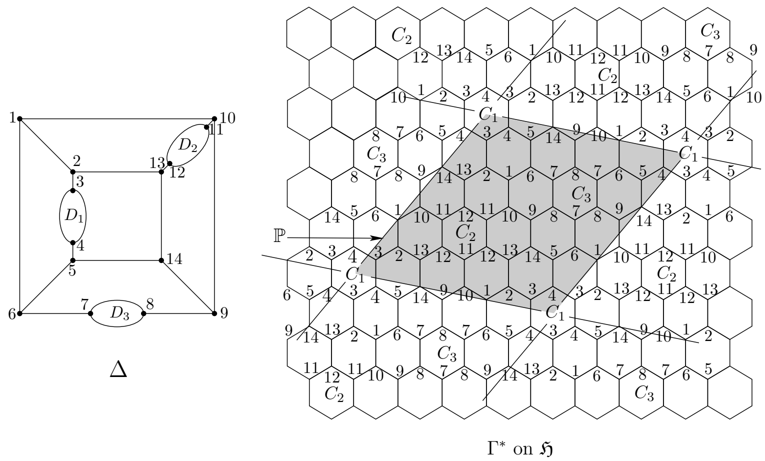


Figure 6. The extended embedding Γ^* of Δ in $\mathfrak{H}$

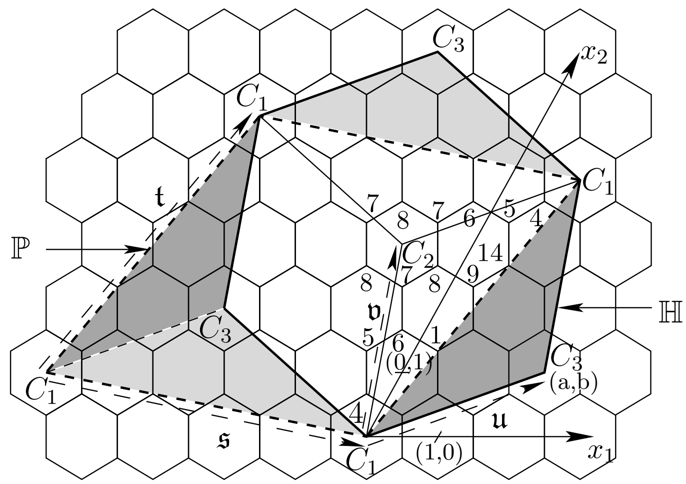
denote this infinite representation. Graph $\mathbb{D}$ is retrieved from Γ^* by identifying all vertices with the same label.

Γ^* possesses two kinds of symmetry, translational and rotational. The translational symmetry is given by the twofold periodicity. For the rotational symmetry, consider any face (hexagon) $\mathbf{H}$ of Γ^* that corresponds to a digon with vertices v_i, v_j, say: then the vertices of $\mathbf{H}$ are cyclically labelled i, j, i, j, i, j. Any rotation of Γ^* around the centre of $\mathbf{H}$ by a multiple of 120 degrees preserves the labelling of $\mathbf{H}$ and, because of the uniqueness of the continuation of the labelling from $\mathbf{H}$ via Δ and Γ to Γ^*, also preserves the labelling of the whole of Γ^*. As a consequence, $\mathbb{P}$ may be chosen to be a 60/120 degrees rhombus (Figure 6).

From $\mathbb{P}$ we obtain another representation of Γ by a regular hexagon $\mathbb{H}$ over Γ^* (Figure 7(a)) with boundary identification as indicated in Figure 7(b).

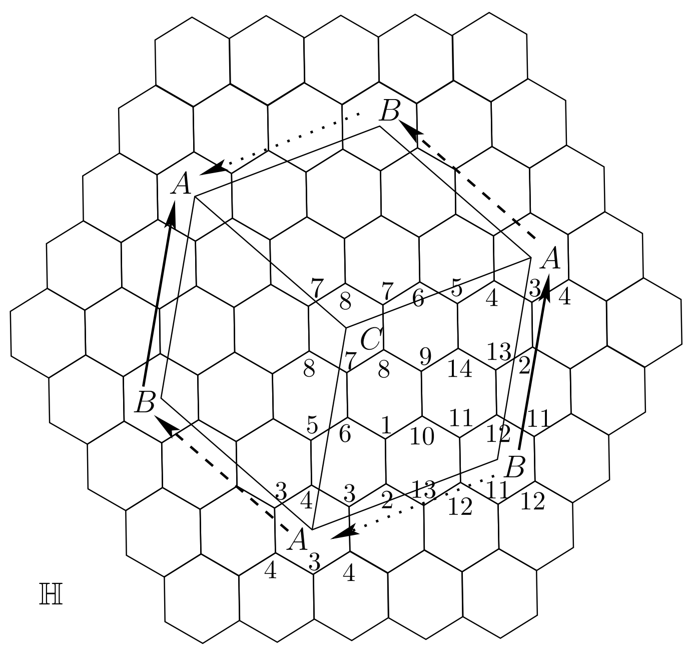
As depicted in Figure 7(b), by rays issuing from its centre, $\mathbb{H}$ is dissected into three congruent 60/120 degrees rhombi each of which, with boundary identification as given by the arrows in Figure 8, carries a labelled representation Δ' of the $(2, 6)$-cage Δ. Gluing together corresponding parts of the boundary of Δ' we retrieve Δ in a specific spatial shape (Figure 9).

As described in Figures 7 and 10, we introduce 60 degrees cartesian coordinates x_1, x_2 in the plane of $\mathbb{H}$ (i.e., in $\mathfrak{H}$) such that point A becomes the origin and the coordinates a, b of point B are rational integers satisfying $a > 0$, $b \geq 0$ (this last condition can always be satisfied, it determines the directions of the axes). Note



(a) $\mathfrak{u} = (a, b)$ $\quad$ $\mathfrak{s} = 2\mathfrak{u} - \mathfrak{v} = (2a + b, b - a)$

$\mathfrak{v} = (-b, a + b)$ $\quad$ $\mathfrak{t} = \mathfrak{u} + \mathfrak{v} = (a - b, 2b + a)$



(b) $A \sim C_1$, $B \sim C_2$, $C \sim C_3$

Figure 7. Transformation of the parallelogram $\mathbb{P}$ into the hexagon $\mathbb{H}$

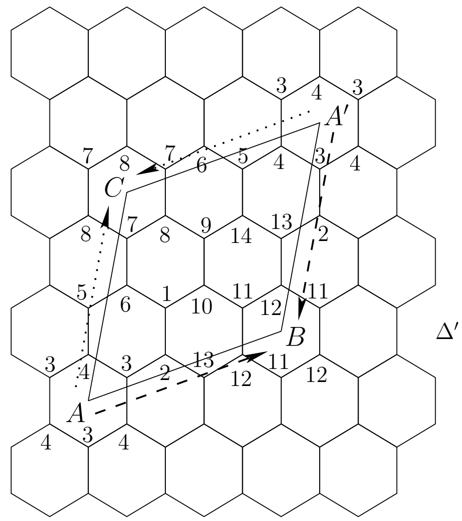


Figure 8. Representation Δ' of Δ

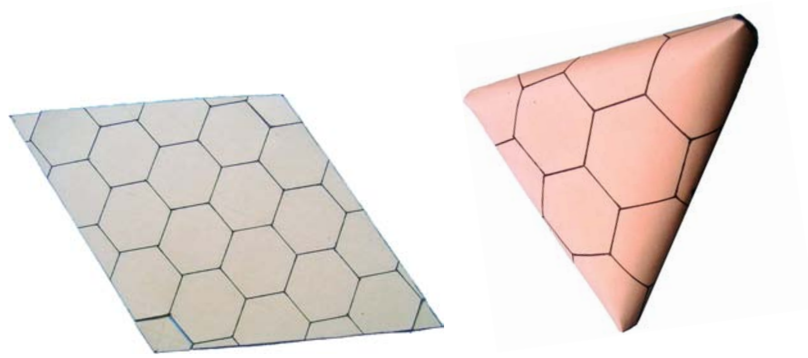


Figure 9. Folding Δ' results in a geometrical (dihedral) representation of Δ

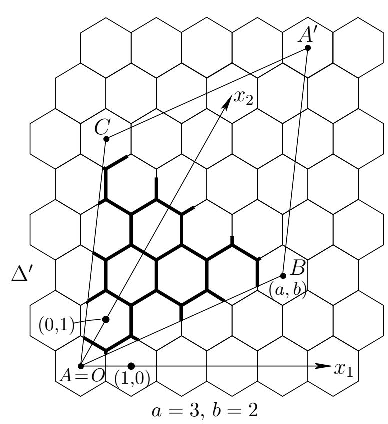


Figure 10. The parameters a, b of a $(2, 6)$-cage Δ

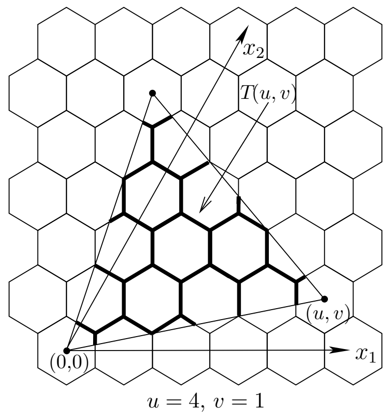


Figure 11. The parameters u, v of a Goldberg transformation

that point C obtaines coordinates $-b, a+b$.

By the above procedure, every $(2,6)$-cage uniquely determines an ordered pair of integers (its parameters) a, b where $a > 0,\ b \geq 0$. The converse is also true: given integers $a > 0,\ b \geq 0$, draw the 60/120 degrees rhombus $\mathbb{R}(a,b)$ spanned by the vectors (a,b) and $(-b, a+b)$ and, by folding and gluing as described above (Figure 9), obtain from $\mathbb{R}(a,b)$ a $(2,6)$-cage with parameters a, b.

Thus we have established a bijection from the set $\mathfrak{D}$ of $(2,6)$-cages onto the set of ordered pairs (a,b) where $a > 0,\ b \geq 0$ are rational integers.

*

The *Goldberg transformation*[b] $G(u,v)$ and the Goldberg transform $\Pi(u,v) = G(u,v)\Pi$ of a cubic pseudo-polyhedron Π are defined as follows.

1) Let $u > 0,\ v \geq 0$ be rational integers and let $T(u,v)$ denote an equilateral triangular section of $\mathfrak{H}$ as depicted in Figure 11 (the bold-face part of Δ' in Figure 10 is $T(a,b)$).

2) Form the dual $\overline{\Pi}$ of Π which is a triangulation of the sphere.

3) Insert into every face of $\overline{\Pi}$ a (duly deformed) copy of $T(u,v)$ and glue these copies together.

The result is a pseudo-polyhedron $\Pi(u,v)$, the Goldberg transform of Π. Note that $\Pi(1,0) = \Pi$.

[b]For the theory of the Goldberg transformation see [2],[4],[8],[12].

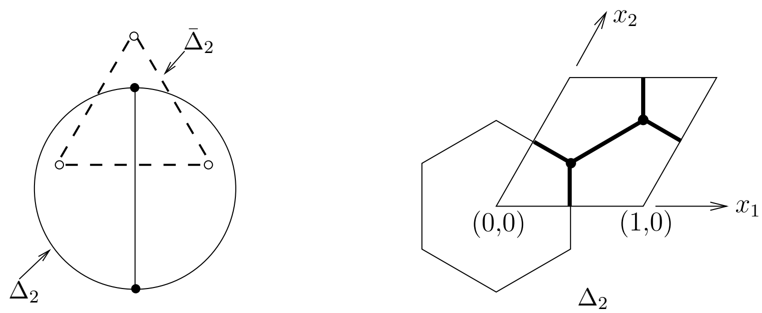


Figure 12. Δ_2 and its dual $\bar{\Delta}_2$ which is a dihedron consisting of two triangular faces

Let Π have $n = 2q$ vertices. Under a Goldberg transformation $\Pi \to \Pi(u, v)$, for $p \neq 6$ the number of p-gons stays the same whereas the number of hexagons increases by $q(u^2 + uv + v^2 - 1)$.

If Π is itself a $(2, 6)$-cage with parameters a, b (Figure 10) then $\Pi(u, v)$ is a $(2, 6)$-cage with parameters

$$\begin{cases} ua - vb, \quad ub + va + vb & \text{if } ua - vb > 0, \\ ua + ub + va, \quad vb - ua & \text{otherwise.} \end{cases} \tag{1}$$

If, in particular, Π is the 2-cage Δ_2 (with $a = 1$, $b = 0$; see Figure 12) then $\Pi(u, v) = \Delta_2(u, v)$ is a $(2, 6)$-cage with parameters u, v (see Figures 10, 11). Thus, with $\mathfrak{G}$ denoting the set of Goldberg transformations, we have

Theorem 1. *The set $\mathfrak{D}$ of $(2, 6)$-cages coincides with the set $\mathfrak{G}\Delta_2$ of Goldberg transforms of the 2-cage Δ_2.*

As a consequence, all $(2, 6)$-cages share the trigonal symmetry of Δ_2 or, equivalently, of its dual $\bar{\Delta}_2$ which is the trigonal dihedron (Figures 12, 9).

I.2 The algebraic structure of the set of $(2, 6)$-cages

Let $\mathfrak{E}$ denote the Eisenstein field, i.e., the quadratic number field generated by $\sqrt{-3}$, and let $\sigma = e^{\frac{2\pi i}{6}} = \frac{1}{2}(1 + i\sqrt{3})$. The powers of σ constitute the 6-membered group $\mathfrak{U}$ of units in $\mathfrak{E}$. The integers in $\mathfrak{E}$ are the numbers $z = x + y\sigma$ where x and y are rational integers. With respect to multiplication, the non-zero integers form a semigroup denoted by $\mathfrak{I}$. Let $\tilde{\mathfrak{I}}$ be the semigroup $\mathfrak{I}/\mathfrak{U}$ whose elements are the cosets $z\mathfrak{U}$ ($z \in \mathfrak{I}$). Every coset $z\mathfrak{U}$ contains precisely one integer $z_0 = x_0 + y_0\sigma$ satisfying

$x_0 > 0$, $y_0 \geq 0$ (i.e., point z_0 lies in the first sextant of the number plane). These integers, representing the elements of $\tilde{\mathfrak{I}}$, define a set $\mathfrak{P}$ which may be called the set of *positive* integers of $\mathfrak{E}$. Let $q, r, s \in \mathfrak{P}$ and define composition $\circ$ by $s = q \circ r$ *if and only if* $\dfrac{qr}{s} \in \mathfrak{U}$. Under this composition, $\mathfrak{P}$ is an abelian semigroup isomorphic to $\tilde{\mathfrak{I}}$.

Let $\mathfrak{G}$ be the set of Goldberg transformations $G = G(u, v)$ (Section I.1). Consider the bijection from $\mathfrak{G}$ onto $\mathfrak{P}$ induced by the correspondence

$$G = G(u, v) \leftrightarrow w = u + v\sigma; \quad \text{write, briefly,} \quad G(u, v) = G(w).$$

It is well known [12] that this bijection is composition preserving, i.e., if compositum $G_1 \,\Box\, G_2$ is defined by performing G_1 after G_2 then

$$G(w_1) \,\Box\, G(w_2) = G(w_1 \circ w_2)$$

(implying, in particular, that the composition of Goldberg transformations is commutative and associative).

Let $\mathfrak{D}$ be the set of $(2, 6)$-cages. For $\Delta \in \mathfrak{D}$ with parameters a, b briefly write

$$\Delta(a, b) = \Delta(c) \quad \text{where} \quad c = a + b\sigma \in \mathfrak{P}.$$

We use the bijection from $\mathfrak{D}$ onto $\mathfrak{G}$ given by the formula $\Delta(c) = G(c)\Delta_2$ (Section I.1) to define a composition $\Diamond$ of $(2, 6)$-cages. We have

$$G(c_1)\Delta(c_2) = G(c_1)(G(c_2)\Delta_2) = (G(c_1) \,\Box\, G(c_2))\Delta_2 = G(c_1 \circ c_2)\Delta_2$$

and define

$$\Delta(c_1) \Diamond \Delta(c_2) = G(c_1)\Delta(c_2) \quad \text{implying} \quad \Delta(c_1) \Diamond \Delta(c_2) = \Delta(c_1 \circ c_2).$$

Thus the algebraic-number theoretic structure of $\mathfrak{P}$ is transferred from $\mathfrak{P}$ via $\mathfrak{G}$ onto $\mathfrak{D}$.

$(2, 6)$-cage Δ' is a *factor* (to be distinguished from a divisor) of $(2, 6)$-cage Δ (in symbols: $\Delta' \underset{F}{|} \Delta$) if and only if there is a $(2, 6)$-cage Δ'' such that $\Delta' \Diamond \Delta'' = \Delta$. Cage $\Delta_2 = \Delta(1)$ acts as the unique unity, and every $(2, 6)$-cage different from Δ_2 has a decomposition into prime factors (called *prime cages*) which is unique up to the arrangement of the factors. Δ is a prime cage if and only if $\Delta = \Delta(p)$ where p is a positive prime number in $\mathfrak{E}$.

Using the above definition of the composition of $(2,6)$-cages it is easy to see that every factor is also a divisor:

Theorem 2. $\Delta' \underset{F}{|} \Delta$ *implies* $\Delta' \mid \Delta$.

Remark. The question whether the converse of Theorem 2 is also true in the sense that if both Δ and Δ' are $(2,6)$-cages and Δ' is a divisor of Δ then it is also a factor of Δ, remains open.

Let $f_{\mathbb{G}}(\lambda)$ denote the characteristic polynomial of the graph $\mathbb{G}$ (or its embedding). As is well known, $\mathbb{G}' \mid \mathbb{G}$ implies $f_{\mathbb{G}'}(\lambda) \mid f_{\mathbb{G}}(\lambda)$ ([5]), thus we have the following Corollary to Theorem 2.

Corollary 1. $\Delta' \underset{F}{|} \Delta$ *implies* $f_{\Delta'}(\lambda) \mid f_{\Delta}(\lambda)$ *i.e.* $\mathrm{Spec}(\Delta') \subseteq \mathrm{Spec}(\Delta)$.
See also Theorem 4.

Part II. Spectrum and eigenvectors of a $(2,6)$-cage

II.1 An intermediary step: The spectrum of Γ

We shall first calculate spectrum and eigenvectors of the toroidal 6-cage $\Gamma = \Gamma(\Delta)$ associated with Δ.

*

From [9] we take the

General Theorem. *Let $\mathbb{S}$ be any toroidal 6-cage.*
We shall refer to a fixed representation Σ of $\mathbb{S}$ with parallelogram $\mathcal{P}$ spanned by vectors $(a_1, a_2), (b_1, b_2)$, with vertex coordinates $(x_1, x_2; j)$, as depicted in Figure 13.
The vertex set of $\mathbb{S}$ is

$$V(\mathbb{S}) = \{(x_1, x_2; j) \mid \textit{point } (x_1, x_2; j) \textit{ belongs to } \mathcal{P}\}.$$

Let $d = |a_1 b_2 - a_2 b_1|$. The equations

$$\varrho_1^{a_1} \varrho_2^{a_2} = \varrho_1^{b_1} \varrho_2^{b_2} = 1, \tag{2}$$

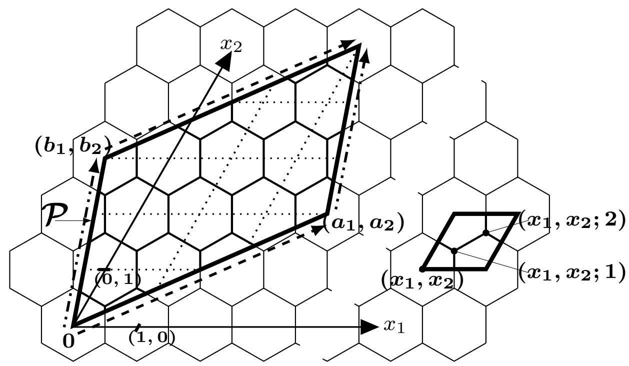


Figure 13. Parallelogram representation Σ of the toroidal 6-cage $\mathbb{S}$. The arrows (- - ->, $\cdots - \cdots$->) indicate the boundary identification of $\mathcal{P}$.

called the key equations*, have precisely d distinct solutions (the* keys*) $\boldsymbol{\varrho} = (\varrho_1, \varrho_2)$ forming the key set $\mathfrak{R}(\mathbb{S})$ which does not depend on the special parallelogram representation of $\mathbb{S}$. The components ϱ_1, ϱ_2 of any key are d^{th} roots of unity.*

Clearly, with $\boldsymbol{\varrho} = (\varrho_1, \varrho_2)$ also $\bar{\boldsymbol{\varrho}} = (\bar{\varrho}_1, \bar{\varrho}_2)$ is a key. The keys can easily be computed by means of an elementary algorithm (see the Appendix) which is essentially identical with Euclid's algorithm for determining the greatest common divisor of two integers. In what follows we assume the keys to be known.

Let

$$\lambda(\mu; \boldsymbol{\varrho}) = (-1)^{\mu-1}\,|1 + \varrho_1 + \varrho_2|, \tag{3}$$

and let (in accordance with Chapter II.1, equation (7), in [9]) $u_1(\mu; \boldsymbol{\varrho})$ *and* $u_2(\mu; \boldsymbol{\varrho})$ *satisfy the equations*

$$(1 + \overline{\varrho_1} + \overline{\varrho_2})u_2(\mu; \boldsymbol{\varrho}) = \lambda(\mu; \boldsymbol{\varrho}) \cdot u_1(\mu; \boldsymbol{\varrho}), \quad |u_1(\mu; \boldsymbol{\varrho})| = |u_2(\mu; \boldsymbol{\varrho})| = 1$$

$$(\boldsymbol{\varrho} \in \mathfrak{R}(\mathbb{S}), \quad \mu = 1, 2).$$

Then the numbers $\lambda(\mu; \boldsymbol{\varrho})$ and vectors

$$\mathbf{e}(\mu; \boldsymbol{\varrho}) = \{\varrho_1^{x_1}\varrho_2^{x_2}u_j(\mu; \boldsymbol{\varrho}) \mid (x_1, x_2; j) \in V(\mathbb{S})\} \tag{4}$$

form the spectrum and a corresponding eigenvector system of $\mathbb{S}$, the vectors $\mathbf{e}(\mu, \boldsymbol{\varrho})$ being pairwise orthogonal with common norm $2d$.

Zero is an eigenvalue of $\mathbb{S}$, necessarily of multiplicity 4, if and only if $a_1 - a_2 \equiv b_1 - b_2 \equiv 0$, mod 3.

*

Next we apply this theorem to the special case $\mathbb{S} = \Gamma = \Gamma(\Delta)$ where the parallelogram $\mathcal{P}$ is the 60/120 degrees rhombus $\mathbb{P}$ (Figures 6, 7) spanned by the vectors

$$\mathfrak{s} = (a_1, a_2) = (2a + b, b - a), \qquad \mathfrak{t} = (b_1, b_2) = (a - b, 2b + a),$$

see Figure 7. Note that

$$d = \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} = 3(a^2 + ab + b^2).$$

The key equations reduce to

$$\varrho_1^{2a+b} \varrho_2^{b-a} = \varrho_1^{a-b} \varrho_2^{2b+a} = 1; \tag{5}$$

for $\mathfrak{R}(\Gamma)$ we shall also write $\mathfrak{R}(a, b)$.

If C denotes a collection of elements, with repetitions allowed, then let C^* denote the corresponding set.

Consider the collections

$$\begin{cases} K &= \{(\varrho_1, \varrho_2), (\bar\varrho_1 \varrho_2, \bar\varrho_1), (\bar\varrho_2, \varrho_1 \bar\varrho_2)\}, \\ \bar K &= \{(\bar\varrho_1, \bar\varrho_2), (\varrho_1 \bar\varrho_2, \varrho_1), (\varrho_2, \bar\varrho_1 \varrho_2)\}, \end{cases} \qquad \mathfrak{K} = K \cup \bar K; \tag{6}$$

note that K is closed under the transformation

$$Q: \quad (\varrho_1, \varrho_2) \longrightarrow (\bar\varrho_1 \varrho_2, \bar\varrho_1). \tag{7}$$

*

The following 10 propositions can easily be verified.

(i) *If one element of $\mathfrak{K}$ is a key then all elements of $\mathfrak{K}$ are keys.*

From now on consider all and only such collections $\mathfrak{K}$ that contain a key of Γ. The sets K^* (and $\bar K^*$) and $\mathfrak{K}^*$ are called the *key families* and *clans* of Γ, respectively.

(ii) *If two key clans have a key in common then they are identical.*

(iii) $\{(1, 1)\}$ *is a key family as well as a key clan* (the trivial one).

(iv) *If both a and b are even then $\{(1, -1), (-1, 1), (-1, -1)\}$ is a key family as well as a key clan; otherwise none of $(1, -1)$, $(-1, 1)$, $(-1, -1)$ is a key.*

Let $\varepsilon = e^{\frac{1}{3}2\pi i} = -\frac{1}{2} + \frac{i}{2}\sqrt{3}$.

(*v*) $\{(\varepsilon, \bar{\varepsilon})\}, \{(\bar{\varepsilon}, \varepsilon)\}$ *are key families,* $\{(\varepsilon, \bar{\varepsilon}), (\bar{\varepsilon}, \varepsilon)\}$ *is a key clan.*

(*vi*) *If* $\mathfrak{K}$ *contains a non-real key distinct from* $(\varepsilon, \bar{\varepsilon})$ *and* $(\bar{\varepsilon}, \varepsilon)$ *then all keys in* $\mathfrak{K}$ *are non-real and pairwise distinct, hence, because of* (*ii*) *and* (*v*)*, also distinct from* $(\varepsilon, \bar{\varepsilon})$ *and* $(\bar{\varepsilon}, \varepsilon)$*; thus* $\mathfrak{K}$ *is a key clan.*

Statements (*i*)–(*vi*) imply the following proposition.

$\mathfrak{R}(a, b)$ *is the disjoint union of the key clans*

$\{(1, 1)\}$,

$\{(1, -1), (-1, 1), (-1, -1)\}$ *which is present if and only if both* a *and* b *are even,*

$\{(\varepsilon, \bar{\varepsilon}), (\bar{\varepsilon}, \varepsilon)\} \quad (\varepsilon = e^{\frac{1}{3}2\pi i})$,

and

a set of disjoint key clans, each of cardinality 6*, which consist of non-real keys only.* This last set comprising all 6-membered clans will be denoted by $\mathfrak{C}(a, b)$.

Define $\qquad \varphi(\boldsymbol{\varrho}) = \varphi(\varrho_1, \varrho_2) = |1 + \varrho_1 + \varrho_2|$.

(*vii*) $\varphi(\boldsymbol{\varrho})$ *takes on the same value for all members of a key clan.*

(*viii*) $\varphi(1, 1) = 3$.

(*ix*) $\varphi(1, -1) = \varphi(-1, 1) = \varphi(-1, -1) = 1$ (relevant only if $a \equiv b \equiv 0$, mod 2).

(*x*) $\varphi(\boldsymbol{\varrho}) = 0$ *if and only if* $\boldsymbol{\varrho} = (\varepsilon, \bar{\varepsilon})$ *or* $\boldsymbol{\varrho} = (\bar{\varepsilon}, \varepsilon)$ (cf. [9]).

*

Arbitrarily select one member of each of the clans in $\mathfrak{C}(a, b)$ and let $\mathfrak{K}_C(a, b)$ denote the set of these representatives. Analogously, let $\mathfrak{K}_F(a, b)$ be a key set that contains precisely one key from each 3-membered key family of Γ; we may assume that with $\varrho \in \mathfrak{K}_F(a, b)$ also $\bar{\varrho} \in \mathfrak{K}_F(a, b)$.

Applying the General Theorem we can now write down spectrum and eigenvectors of (the graph $\mathbb{G}$) of Γ, the "toroidal parent" of Δ. In particular, the characteristic polynomial of Γ is

$$f_\Gamma(\lambda) = (\lambda^2 - 9) \cdot \lambda^4 \cdot (\lambda^2 - 1)^{3\gamma} \cdot \prod_{\boldsymbol{\varrho} \in \mathfrak{K}_C(a,b)} (\lambda^2 - |1 + \varrho_1 + \varrho_2|^2)^6$$

where $\gamma = 1$ if $a \equiv b \equiv 0$, mod 2 and $\gamma = 0$ otherwise.

II.2 Spectrum and eigenvectors of Δ

If vectors

$$\mathbf{c} = \{c(v) \mid v \in V(\Gamma)\} \quad \text{and} \quad \mathbf{c^*} = \{c^*(v) \mid v \in V(\Delta)\}$$

satisfy

$$c(v) = c^*(v_i) \quad \text{for the three vertices } v \text{ of } \Gamma \text{ labelled } i \ (i = 1, 2, \ldots, n) \tag{8}$$

then $\mathbf{c}$ is called the *extension* of $\mathbf{c^*}$ from Δ to Γ, and $\mathbf{c}^*$ is called the *restriction* of $\mathbf{c}$ from Γ to Δ.

Let vector $\mathbf{e}$ on Γ be the extension of vector $\mathbf{e}^*$ on Δ. If $\mathbf{e}^*$ is an eigenvector of Δ (with eigenvalue λ) then – by the fact that graph $\mathbb{D}$ is a divisor of graph $\mathbb{G}$ (i.e., graph $\mathbb{G}$ is a cover of graph $\mathbb{D}$) – $\mathbf{e}$ is an eigenvector of Γ (with eigenvalue λ), and conversely.

Given an eigenvector $\mathbf{e}^*$ of Δ, its extension $\mathbf{e}$ to Γ, as an eigenvector of Γ, is a linear combination of some of those eigenvectors of Γ specified by the General Theorem (formula (4)). By constructing these linear combinations and restricting them to Δ we shall retrieve the eigenvectors and eigenvalues of Δ from those of Γ.

*

First consider the trivial key $(1, 1)$ of Γ which, by (3) and (4), gives rise to eigenvalues

$$\lambda(1; 1, 1) = 3, \quad \lambda(2; 1, 1) = -3 \tag{9}$$

and, with $u_1(\mu; 1, 1) = (-1)^\mu u_2(\mu; 1, 1) = 1$, to the corresponding eigenvector components

$$e(1; 1, 1 \mid x_1, x_2; j) = 1, \quad e(2; 1, 1 \mid x_1, x_2; j) = (-1)^{\,j} \quad (j = 1, 2) \tag{10}$$

which immediately implies that the eigenvectors $\mathbf{e}(1; 1, 1)$ and $\mathbf{e}(2; 1, 1)$ themselves can be restricted from Γ to Δ, and that their restrictions are eigenvectors of Δ with eigenvalues 3, -3, respectively. It is convenient to use

$$\tilde{\mathbf{e}}(1; 1, 1) = \sqrt{3}\, \mathbf{e}(1; 1, 1) \quad \text{and} \quad \tilde{\mathbf{e}}(2; 1, 1) = \sqrt{3}\, \mathbf{e}(2; 1, 1) \tag{11}$$

instead: their restrictions $\mathbf{e^*}$ from Γ to Δ are real orthogonal eigenvectors of Δ with common norm $\frac{1}{3} \cdot 3 \cdot |V(\Gamma)| = 2d$.

Next let key $\boldsymbol{\varrho}$ of Γ be distinct from $(1,1)$, $(\varepsilon,\bar{\varepsilon})$, $(\bar{\varepsilon},\varepsilon)$. Then the three keys

$$\boldsymbol{\varrho}=(\varrho_1,\varrho_2),\quad \boldsymbol{\varrho}'=(\bar{\varrho}_1\varrho_2,\bar{\varrho}_1),\quad \boldsymbol{\varrho}''=(\bar{\varrho}_2,\varrho_1\bar{\varrho}_2)$$

are pairwise distinct and form a family with eigenvalues

$$\lambda(\mu;\boldsymbol{\varrho})=\lambda(\mu;\boldsymbol{\varrho}')=\lambda(\mu;\boldsymbol{\varrho}'')=(-1)^{\mu-1}|1+\varrho_1+\varrho_2|\neq 0\quad(\mu=1,2). \tag{12}$$

The eigenvector $\mathbf{e}(\mu;\boldsymbol{\varrho})$ of Γ has components $e(\mu;\boldsymbol{\varrho}\mid\mathbf{x};j)=\varrho_1^{x_1}\varrho_2^{x_2}u_j(\mu;\boldsymbol{\varrho})$. We also need $\mathbf{e}(\mu;\boldsymbol{\varrho}')$ and $\mathbf{e}(\mu;\boldsymbol{\varrho}'')$. Because of

$$\begin{aligned}1+\overline{\varrho_1'}+\overline{\varrho_2'}&=\varrho_1(1+\overline{\varrho_1}+\overline{\varrho_2}),\\ 1+\overline{\varrho_1''}+\overline{\varrho_2''}&=\varrho_2(1+\overline{\varrho_1}+\overline{\varrho_2})\end{aligned}$$

and

$$\begin{aligned}&\lambda(\mu;\boldsymbol{\varrho})=\lambda(\mu;\boldsymbol{\varrho}')=\lambda(\mu;\boldsymbol{\varrho}'')=(-1)^{\mu}|1+\varrho_1+\varrho_2|,\\ &(1+\overline{\varrho_1}+\overline{\varrho_2})u_2(\mu;\boldsymbol{\varrho})=\lambda(\mu;\boldsymbol{\varrho})u_1(\mu;\boldsymbol{\varrho})\end{aligned}$$

we may choose

$$\begin{aligned}u_j(\mu;\boldsymbol{\varrho}')&=\sqrt{\varrho_1}^{\,3-2j}u_j(\mu;\boldsymbol{\varrho})^{\mathrm{c}},\\ u_j(\mu;\boldsymbol{\varrho}'')&=\sqrt{\varrho_2}^{\,3-2j}u_j(\mu;\boldsymbol{\varrho})\qquad(\mu=1,2;j=1,2).\end{aligned}$$

This yields

$$\begin{aligned}e(\mu;\boldsymbol{\varrho}'\mid\mathbf{x};j)&=(\overline{\varrho_1}\varrho_2)^{x_1}\overline{\varrho_1}^{x_2}\sqrt{\varrho_1}^{\,3-2j}u_j(\mu;\boldsymbol{\varrho})=\varrho_1^{-x_1-x_2-j+1}\varrho_2^{x_1}\sqrt{\varrho_1}u_j(\mu;\boldsymbol{\varrho}),\\ e(\mu;\boldsymbol{\varrho}''\mid\mathbf{x};j)&=\overline{\varrho_2}^{x_1}(\overline{\varrho_2}\varrho_1)^{x_2}\sqrt{\varrho_2}^{\,3-2j}u_j(\mu;\boldsymbol{\varrho})=\varrho_1^{x_2}\varrho_2^{-x_1-x_2-j+1}\sqrt{\varrho_2}u_j(\mu;\boldsymbol{\varrho}).\end{aligned}\tag{13}$$

Consider the linear combination

$$\tilde{\mathbf{e}}(\mu;\boldsymbol{\varrho})=c\,\mathbf{e}(\mu;\boldsymbol{\varrho})+c'\mathbf{e}(\mu;\boldsymbol{\varrho}')+c''\mathbf{e}(\mu;\boldsymbol{\varrho}'')$$

with numbers c,c',c'', not all equal to zero.

$\mathbf{e}(\mu;\boldsymbol{\varrho})$, $\mathbf{e}(\mu;\boldsymbol{\varrho}')$, $\mathbf{e}(\mu;\boldsymbol{\varrho}'')$ being linearly independent eigenvectors of Γ belonging to the same eigenvalue $\lambda(\mu;\boldsymbol{\varrho})$, vector $\tilde{\mathbf{e}}(\mu;\boldsymbol{\varrho})$, too, is an eigenvector of Γ belonging

[c] For $z\neq 0$ we assume $0\leq \arg\sqrt{z}<\pi$.

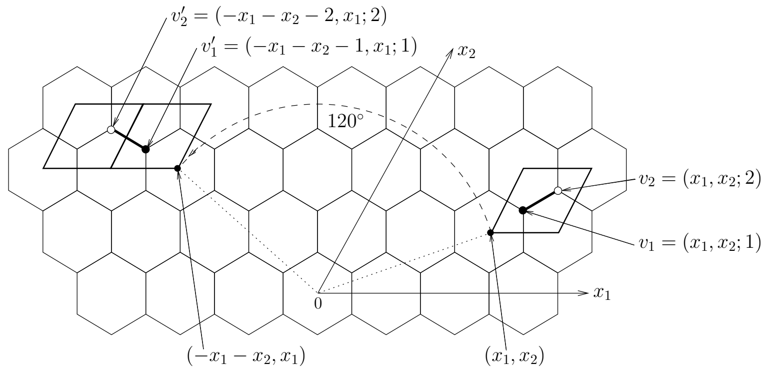


Figure 14. $T:\ v = (x_1, x_2; j) \to v' = (-x_1 - x_2 - j, x_1; j)$

to the eigenvalue $\lambda(\mu; \boldsymbol{\varrho})$. By (13), the components of $\tilde{\mathbf{e}}(\mu; \boldsymbol{\varrho})$ are

$$\begin{aligned} \tilde{e}(\mu; \boldsymbol{\varrho} \mid \mathbf{x}; j) &= c\, e(\mu; \boldsymbol{\varrho} \mid \mathbf{x}; j) + c'\, e(\mu; \boldsymbol{\varrho'} \mid \mathbf{x}; j) + c''\, e(\mu; \boldsymbol{\varrho''} \mid \mathbf{x}; j) \\ &= (c\varrho_1^{x_1} \varrho_2^{x_2} + c'\, \varrho_1^{-x_1-x_2-j+1} \varrho_2^{x_1} \sqrt{\varrho_1} \\ &\quad + c''\varrho_1^{x_2} \varrho_2^{-x_1-x_2-j+1} \sqrt{\varrho_2}) u_j(\mu; \boldsymbol{\varrho}). \end{aligned} \tag{14}$$

Let $v = (x_1, x_2; j)$, v' and v'' be the three vertices of Γ corresponding to the same vertex of Δ (Figure 14). If $\tilde{\mathbf{e}}(\mu; \boldsymbol{\varrho})$ is required to be the extension of some vector on Δ then the coefficients c, c', c'' must satisfy the conditions

$$\tilde{e}(\mu; \boldsymbol{\varrho} \mid v) = \tilde{e}(\mu; \boldsymbol{\varrho} \mid v') = \tilde{e}(\mu; \boldsymbol{\varrho} \mid v''). \tag{15}$$

Conversely: (15) implies that $\tilde{\mathbf{e}}(\mu; \boldsymbol{\varrho})$ can be restricted from Γ to Δ.

Vertex v' is obtained from vertex v by a rotation of 120 degrees (Figure 14), i.e., expressed in coordinates $(x_1, x_2; j)$, by the transformation

$$\mathsf{T}: \quad (x_1, x_2; j) \longrightarrow (-x_1 - x_2 - j, x_1; j), \tag{16}$$

thus

$$\begin{gathered} v = (x_1, x_2; j),\ \mathsf{T}v = v' = (-x_1 - x_2 - j, x_1; j), \\ \mathsf{T}v' = v'' = (x_2, -x_1 - x_2 - j; j); \quad \mathsf{T}v'' = v. \end{gathered}$$

Inserting these points into (15) and using (14) we obtain the condition

$$\begin{array}{lllll} & c \cdot \varrho_1^{x_1} \varrho_2^{x_2} & +c' \cdot \varrho_1^{-x_1-x_2-j+1} \varrho_2^{x_1} \sqrt{\varrho_1} & +c'' \cdot \varrho_1^{x_2} \varrho_2^{-x_1-x_2-j+1} \sqrt{\varrho_2} \\ = & c \cdot \varrho_1^{-x_1-x_2-j} \varrho_2^{x_1} & +c' \cdot \varrho_1^{x_2+1} \varrho_2^{-x_1-x_2-j} \sqrt{\varrho_1} & +c'' \cdot \varrho_1^{x_1} \varrho_2^{x_2+1} \sqrt{\varrho_2} \\ = & c \cdot \varrho_1^{x_2} \varrho_2^{-x_1-x_2-j} & +c' \cdot \varrho_1^{x_1+1} \varrho_2^{x_2} \sqrt{\varrho_1} & +c'' \cdot \varrho_1^{-x_1-x_2-j} \varrho_2^{x_1+1} \sqrt{\varrho_2} \end{array}$$

which is satisfied by choosing

$$c = \varrho_1 \varrho_2 \sqrt{\varrho_1} \sqrt{\varrho_2}, \quad c' = \varrho_2 \sqrt{\varrho_2}, \quad c'' = \varrho_1 \sqrt{\varrho_1}.$$

This results in a vector $\tilde{\mathbf{e}}_0$ with components

$$\tilde{e}_0(\mu; \boldsymbol{\varrho} \mid \mathbf{x}; j) = \varrho_1 \varrho_2 \sqrt{\varrho_1} \sqrt{\varrho_2} (\varrho_1^{x_1} \varrho_2^{x_2} + \varrho_1^{-x_1-x_2-j} \varrho_2^{x_1} + \varrho_1^{x_2} \varrho_2^{-x_1-x_2-j}) u_j(\mu; \varrho).$$

The factor $\varrho_1 \varrho_2 \sqrt{\varrho_1} \sqrt{\varrho_2}$ being independent of $(\mathbf{x}; j)$, it may be dropped, and by restriction from Γ to Δ we obtain from $\tilde{\mathbf{e}}_0$ an eigenvector $\mathbf{e}^*$ of Δ, namely,

$$\begin{aligned} \mathbf{e}^*(\mu; \boldsymbol{\varrho}) \;=\; & \{(\varrho_1^{x_1} \varrho_2^{x_2} + \varrho_1^{-x_1-x_2-j} \varrho_2^{x_1} \\ & + \varrho_1^{x_2} \varrho_2^{-x_1-x_2-j}) u_j(\mu; \boldsymbol{\varrho}) \mid (x_1, x_2; j) \in V(\Delta)\}. \end{aligned} \tag{17}$$

Thus every key family of Γ, distinct from $\{(\varepsilon, \bar{\varepsilon})\}, \{(\bar{\varepsilon}, \varepsilon)\}$, yields precisely two non-zero eigenvalues and corresponding eigenvectors $\mathbf{e}^*$ for Δ which, as their Γ-parents, are pairwise orthogonal with common norm $\frac{1}{3} \cdot 3 \cdot |V(\Gamma)| = 2d = 3n = 6(a^2 + ab + b^2)$. Their number is $\frac{1}{3}|V(\Gamma)| = |V(\Delta)| = n$, namely, two for each 3-membered key family (formulae (12), (17)), two for the key $(1, 1)$ (formulae (9), (10), (11)), and none for the keys $(\varepsilon, \bar{\varepsilon}), (\bar{\varepsilon}, \varepsilon)$ (the above procedure, applied to $(\varepsilon, \bar{\varepsilon})$ or $(\bar{\varepsilon}, \varepsilon)$, results in the zero vector). Thus, for Δ we have found a complete system of eigenvalues and pairwise orthogonal eigenvectors with common norm $3n$, i.e., we have proved

Theorem 3. *Let Δ be a $(2, 6)$-cage on n vertices with parameters a, b and define for $\mu = 1, 2$*
$\lambda(\mu; 1, 1) = (-1)^{\mu-1} \cdot 3,$
$\mathbf{e}(\mu; 1, 1) = \{(-1)^{(\mu-1)j} \sqrt{3} \mid (x_1, x_2; j) \in V(\Delta)\},$
and, further, for $\varrho \in \mathfrak{K}_F(a, b)$:
$\lambda(\mu; \boldsymbol{\varrho}) = (-1)^{\mu-1} |1 + \varrho_1 + \varrho_2|,$

$$\begin{aligned} \mathbf{e}(\mu; \boldsymbol{\varrho}) \;=\; & \{(\varrho_1^{x_1} \varrho_2^{x_2} + \varrho_1^{-x_1-x_2-j} \varrho_2^{x_1} \\ & + \varrho_1^{x_2} \varrho_2^{-x_1-x_2-j}) u_j(\mu; \boldsymbol{\varrho}) \mid (x_1, x_2; j) \in V(\Delta)\} \end{aligned}$$

where the $u_j(\mu; \boldsymbol{\varrho})$ satisfy
$(1 + \overline{\varrho_1} + \overline{\varrho_2}) u_2(\mu; \boldsymbol{\varrho}) = \lambda(\mu; \boldsymbol{\varrho}) u_1(\mu; \boldsymbol{\varrho}), \quad |u_j(\mu; \boldsymbol{\varrho})| = 1$
(e.g., $u_2(\mu; \boldsymbol{\varrho}) = 1$, $u_1(\mu; \boldsymbol{\varrho}) = (-1)^{\mu-1} \frac{1+\overline{\varrho_1}+\overline{\varrho_2}}{|1+\varrho_1+\varrho_2|}$).

Then the numbers λ and vectors $\mathbf{e}$ form the spectrum and a complete system of eigenvectors for Δ; these eigenvectors are pairwise orthogonal with common norm

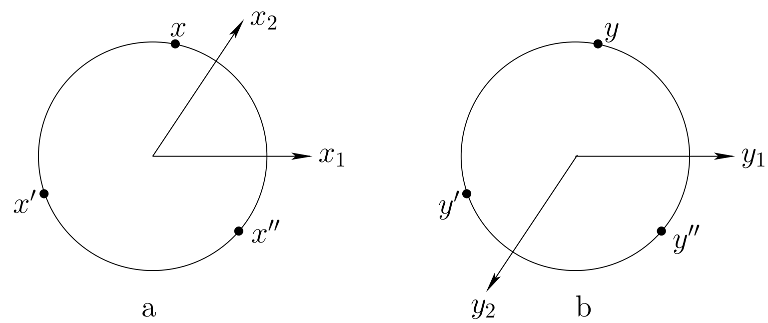


Figure 15. Comparing the transformations of vertices and keys

$3n = 6(a^2 + ab + b^2)$.
Zero is not an eigenvalue.

The characteristic polynomial of Δ *is*

$$f_\Delta(\lambda) = (\lambda^2 - 9)(\lambda^2 - 1)^\delta \prod_{\boldsymbol{\varrho} \in \mathfrak{K}_C(a,b)} (\lambda^2 - |1 + \varrho_1 + \varrho_2|^2)^2$$

where $\delta = 1$ *if* n *is a multiple of* 4 *and* $\delta = 0$ *otherwise.*

Splitting the non-real eigenvectors (which come in pairs $\mathbf{e}, \bar{\mathbf{e}}$) into their real and imaginary part, we also obtain a complete system of real eigenvectors which, again, are pairwise orthogonal.

Remark 1. The transformation of coordinates x_1, x_2 that corresponds to a rotation of 120 degrees (applied twice) is

$$\mathbf{x} = (x_1, x_2) \to \mathbf{x}' = (-x_1 - x_2, x_1) \to \mathbf{x}'' = (x_2, -x_1 - x_2);$$

note that $\quad x_1 + x_1' + x_1'' = x_2 + x_2' + x_2'' = 0 \quad$ (Figure 15a).

Define coordinates $y_1, y_2 \pmod{2\pi}$ of key $\boldsymbol{\varrho} = (\varrho_1, \varrho_2)$ by $\varrho_k = e^{iy_k}$: then, expressed in y_1, y_2, the transformation Q

$$\boldsymbol{\varrho} = (\varrho_1, \varrho_2) \to \boldsymbol{\varrho}' = (\bar{\varrho}_1 \varrho_2, \bar{\varrho}_1)$$

becomes

$$\mathbf{y} = (y_1, y_2) \to \mathbf{y}' = (-y_1 + y_2, -y_1) \to \mathbf{y}'' = (-y_2, y_1 - y_2) \qquad (y_1, y_2 \bmod 2\pi);$$

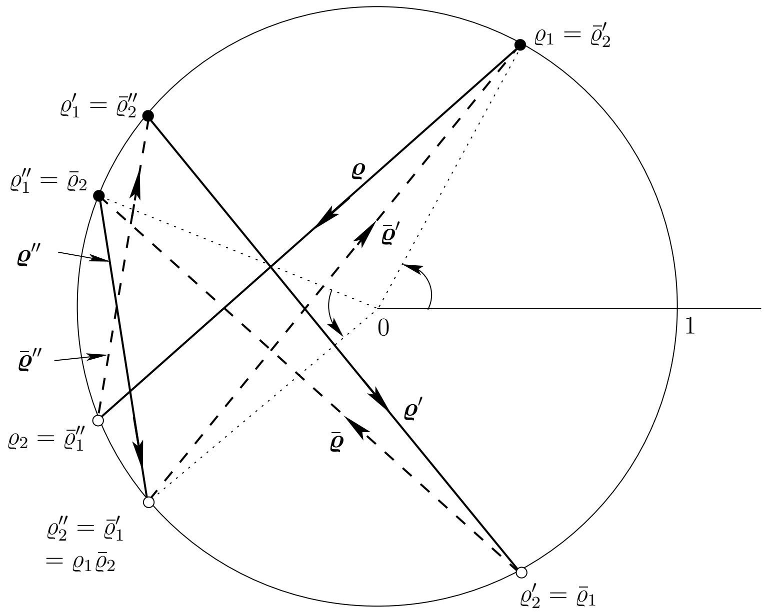


Figure 16. A clan of keys

note that $\varrho_1\varrho_1'\varrho_1'' = \varrho_2\varrho_2'\varrho_2'' = 1$, thus $y_1 + y_1' + y_1'' \equiv y_2 + y_2' + y_2'' \equiv 0$, mod 2π (Figure 15b).

It is the (hidden) similarity of these transformations which lies at the bottom of the theory.

Remark 2. If keys $\boldsymbol{\varrho} = (\varrho_1, \varrho_2)$ are represented by arrows from ϱ_1 to ϱ_2 then the 6-membered clans are represented by pairwise disjoint cyclically orientated self-intersecting hexagons (possibly, degenerate) inscribed in the unit circle (Figure 16).

II.3 The number of perfect matchings in a $(2,6)$-cage

Let B (black), W (white) denote the bipartition of $(2,6)$-cage Δ with parameters a, b; clearly, $|B| = |W|$. If we orientate every edge of Δ from black to white we obtain an edge orientation of Δ with the property that, for every face F of Δ, the number of boundary edges of F whose orientation is in accordance with the positive orientation of the boundary circuit of F is odd (equal to 1 or 3); such an odd edge orientation is sometimes called *Pfaffian* or *Kasteleyn* [10]. Let A and m be the adjacency matrix and the number of perfect matchings of Δ, respectively. Using the above odd orientation, Kasteleyn's theory of counting perfect matchings ([10], see also [11]) immediately yields the equality

$$m^2 = |\det A|.$$

By Theorem 3,

$$|\det A| = |f_\Delta(0)| = 9 \prod_{\boldsymbol{\varrho}\in\mathfrak{K}_C(a,b)} |1+\varrho_1+\varrho_2|^4 \neq 0.$$

Thus we have found

Theorem 4. *Let Δ be a $(2,6)$-cage with parameters a,b. Then the number of perfect matchings of Δ is*

$$m(\Delta) = 3 \prod_{\boldsymbol{\varrho}\in\mathfrak{K}_C(a,b)} |1+\varrho_1+\varrho_2|^2 > 0. \tag{18}$$

In particular: Every $(2,6)$-cage has a perfect matching.

Appendix: *How to calculate the keys.*

This is taken (without proofs) from [9], Section 1.3.

Let a,b be the parameters of some $(2,6)$-cage Δ.

Set $a_1 = 2a+b$, $a_2 = b-a$; $b_1 = a-b$, $b_2 = 2b+a$.

Apply the following algorithm.

Initial matrix: $A = \begin{pmatrix} a_1 & a_2 \\ b_1 & b_2 \end{pmatrix}$.

(I) If $b_1 < 0$: multiply the second row by -1;

(II) if $b_1 = 0$: stop;

(III) if $0 < b_1 < a_1$: subtract the second row from the first;

(IV) if $b_1 \geq a_1$: subtract the first row from the second;

(V) go to (II).

The procedure stops when the transform of A has attained the form $A' = \begin{pmatrix} a'_1 & a'_2 \\ 0 & b'_2 \end{pmatrix}$.

Note that a'_1 is the greatest common divisor of $a_1 = 2a+b$ and $b_1 = b-a$.

The absolute value d of the determinant of A remains unchanged: $d = 3(a^2+ab+b^2) = |a'_1 b'_2|$.

The keys are d^{th} roots of unity.

Set $\alpha = |b'_2|$, $\beta = -a'_2$, $\gamma = a'_1$; $\varepsilon = \exp(2\pi i/d)$ where $d = \alpha\gamma$.

The keys of Δ are: $\boldsymbol{\varrho} = (\varrho_1, \varrho_2)$,

$\varrho_1 = \varepsilon^{\alpha\mu+\beta\nu}$, $\varrho_2 = \varepsilon^{\gamma\nu}$ $(\mu = 0,1,\ldots,\gamma-1; \nu = 0,1,\ldots,\alpha-1)$.

Example:

$a = 4$, $b = 13$: $a_1 = 21$, $a_2 = 9$, $b_1 = -9$, $b_2 = 30$; $d = 711$.

$$A = \begin{pmatrix} 21 & 9 \\ -9 & 30 \end{pmatrix} \xrightarrow{\mathrm{I}} \begin{pmatrix} 21 & 9 \\ 9 & -30 \end{pmatrix} \xrightarrow{2\times\mathrm{III}} \begin{pmatrix} 3 & 69 \\ 9 & -30 \end{pmatrix}$$

$$\xrightarrow{3\times\mathrm{IV}} \begin{pmatrix} 3 & 69 \\ 0 & -237 \end{pmatrix} = A' = \begin{pmatrix} a_1' & a_2' \\ 0 & b_2' \end{pmatrix}.$$

$\alpha = 237$, $\beta = -69$, $\gamma = 3$; $\quad \varepsilon = \exp(2\pi i/711)$.
$\varrho_1 = \varepsilon^{\,237\mu - 69\nu}$, $\varrho_2 = \varepsilon^{3\nu} \quad (\mu = 0, 1, 2\,;\ \nu = 0, 1, \ldots, 236)$.

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